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安定領域はレムニスケート

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1. Introduction

In this paper, we shall consider a periodic system with piecewise constant argument

$$\dot{\mathbf{x}}(t) = p(t)\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\mathbf{x}([t]), \tag{1}$$

where $[\cdot]$ denotes the greatest integer function and $\dot{\mathbf{x}}(t)$ means the right-hand derivative of $\mathbf{x}(t)$ for each integer t. In what follows, we assume the conditions:

(i) $p(t)$ is continuous and ω-periodic on $(-\infty, \infty)$.
(ii) $p(t) = p(\omega/2 - t) = -p(\omega/2 + t)$ for all t.
(iii) $\omega = 4/k$ for some positive integer k.
(iv) $\alpha^2 + \beta^2 > 0$.

Several authors ([1-3]) discussed asymptotic stability for linear delay systems with constant coefficients. For instance ([1]), the zero solution of

$$\dot{\mathbf{x}}(t) = \rho\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{x}(t-\tau)$$

is asymptotically stable if and only if

$$-(\pi/2 - |\theta|) < \rho\tau < 0.$$

However, stability region for periodic delay system

$$\dot{\mathbf{x}}(t) = p(t)\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\mathbf{x}(t-\tau) \tag{2}$$

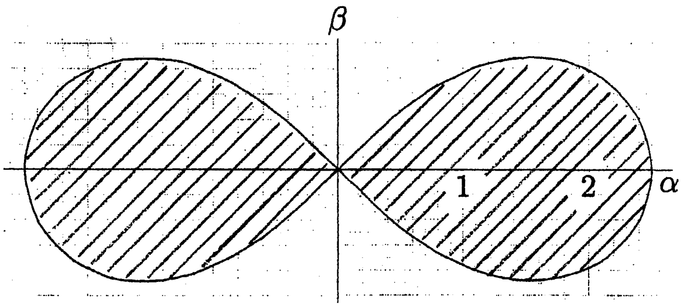
is yet unknown.

For the scalar case, we can find only one result by R. Miyazaki ([2]) which deals with the periodic delay equation

$$\dot{x}(t) = p(t)x(t-\tau), \tag*{(2)$'$}$$

where $p(t)$ satisfies the conditions (a) and (b). Roughly speaking, the zero solution of (2)$'$ is uniformly asymptotically stable if τ is a small positive number. We would like to obtain the stability region for (2). However, this problem is beyond us at the present time.

Recently, using computer, we have made a simulation to find the behavior of solutions for (2). For instance, in the case of $p(t) = \sin(\pi t)$ and $\tau = 1/2$, it seems that stability region for (2) is the interior of lemniscate: $(\alpha^2+\beta^2)^2 = 2a^2(\alpha^2-\beta^2)$ with $a \doteqdot 1.7705$. So, we have the following conjecture.



Conjecture. *The system (2) is uniformly asymptotically stable if and only if the point (α, β) is contained in the interior of some lemniscate.*

This conjecure is still open. But, for the system (1) which is similar to (2) in some sense, we can show the following result corresponding the conjecture.

Theorem 1. *Let $c = \int_0^1 p(t)dt$, and assume that $\int_0^\omega p(t)dt = 0$ and $k \not\equiv 0 \pmod 4$. Then the system (1) is uniformly asymptotically stable if and only if*

$$0 < (\alpha^2+\beta^2)^2 < \frac{2}{c^2}(\alpha^2-\beta^2).$$

2. Main results

Let $r = \sqrt{\alpha^2+\beta^2}$. Then there exists only one $\theta \in (-\pi, \pi]$ such that

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

So, we put

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and $q(t) = rp(t)$. Then the system (1) is reduced to

$$\dot{\mathbf{x}}(t) = q(t)R(\theta)\mathbf{x}([t]), \tag{3}$$

where $q(t)$ satisfies the same properties as $p(t)$:

(a)′ $q(t)$ is continuous and ω-periodic on $(-\infty, \infty)$.

(b)′ $q(t) = q(\omega/2 - t) = -q(\omega/2 + t)$ for all t.

The properties (a) through (c) ensure the following lemma.

Lemma 1. *Assume $\int_0^\omega q(t)dt = 0$. If k is odd, then*

$$\int_0^1 q(t)dt = \int_1^2 q(t)dt = -\int_2^3 q(t)dt = -\int_3^4 q(t)dt.$$

If $k = 4m + 2$ for some integer $m \geq 0$, then

$$\int_0^1 q(t)dt = -\int_1^2 q(t)dt.$$

Proof. Let $k = 4m \pm 1$ for some integer m. Since $\omega = 4/k$, we have

$$m\omega \pm \omega/4 = 1.$$

Hence periodicity of $q(t)$ implies

$$\int_0^1 q(t)dt = \int_0^{m\omega} q(t)dt + \int_{m\omega}^{m\omega \pm \omega/4} q(t)dt = \int_0^{\pm\omega/4} q(t)dt,$$

and also

$$\int_0^2 q(t)dt = \int_0^{\pm\omega/2} q(t)dt.$$

It follows from (b)′ that

$$\int_0^{\pm\omega/2} q(t)dt = 2\int_0^{\pm\omega/4} q(t)dt.$$

This implies

$$\int_0^2 q(t)dt = 2\int_0^1 q(t)dt.$$

Therefore we arrive at

$$\int_0^1 q(t)dt = \int_1^2 q(t)dt.$$

On the other hand, if $k = 4m + 2$, then $2 = (2m + 1)\omega$ and so

$$\int_0^2 q(t)dt = \int_0^{2(m+1)\omega} q(t)dt = 0,$$

which implies

$$\int_1^2 q(t)dt = -\int_0^1 q(t)dt.$$

We can also prove the other equalities. □

Let $\mathbf{x}(t)$ be a solution of (3). It is convenient to denote $\|\mathbf{x}(n)\|$ by ρ_n for each integer n. There exists only one $\varphi \in [0, 2\pi)$ such that

$$\mathbf{x}(n) = R(\varphi)\begin{pmatrix}\rho_n\\ 0\end{pmatrix}.$$

So, we put $\mathbf{u}_n(t) = R(-(\theta+\varphi))\mathbf{x}(t)$ for every n. Then $\mathbf{u}_n(t)$ satisfies

$$\dot{\mathbf{u}}_n(t) = q(t)\begin{pmatrix}\rho_n\\ 0\end{pmatrix}, \quad \mathbf{u}_n(n) = \rho_n\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix}$$

for $t \in [n, n+1)$. Hence it follows that

$$\mathbf{u}_n(t) = \rho_n\begin{pmatrix}\cos\theta + \int_n^t q(s)ds\\ -\sin\theta\end{pmatrix} \tag{4}$$

for $t \in [n, n+1]$. Now we give a result for the case $k \not\equiv 0 \pmod 4$.

Proposition 1. *Let $\gamma = \int_0^1 q(t)dt$, and assume $\int_0^\omega q(t)dt = 0$ and $k \not\equiv 0$ (mod 4). Then the system* (3) *is uniformly asymptotically stable if and only if*

$$0 < |\gamma| < \sqrt{2\cos 2\theta}. \tag{5}$$

Proof. First, we consider the case of $k = 4m \pm 1$. It follows from (4) that

$$\rho_{n+1}^2 = \rho_n^2\left\{1 + 2\cos\theta\int_n^{n+1} q(s)ds + \left(\int_n^{n+1} q(s)ds\right)^2\right\}.$$

Lemma 1 implies

$$\int_n^{n+1} q(s)ds = \begin{cases}\gamma & \text{if } n = 0, 1 \pmod 4\\ -\gamma & \text{if } n = 2, 3 \pmod 4).\end{cases}$$

Hence we have

$$\rho_{n+1}^2 = \begin{cases}\rho_n^2(1 + 2\gamma\cos\theta + \gamma^2) & \text{if } n = 0, 1 \pmod 4\\ \rho_n^2(1 - 2\gamma\cos\theta + \gamma^2) & \text{if } n = 2, 3 \pmod 4),\end{cases}$$

and so

$$\begin{aligned}\rho_{n+4}^2 &= \rho_n^2(1 + 2\gamma\cos\theta + \gamma^2)^2(1 - 2\gamma\cos\theta + \gamma^2)^2\\ &= \rho_n^2(1 - 2\gamma^2\cos 2\theta + \gamma^4)^2\end{aligned}$$

for each n. Thus the ratio ρ_{n+4}/ρ_n is independent of $\mathbf{x}(t)$ and n. It is easy to see that $\rho_{n+4}/\rho_n < 1$ if and only if

$$\gamma \neq 0 \quad \text{and} \quad \gamma^2 < 2\cos 2\theta,$$

which is equivalent to (5). Next consider the case of $k = 4m + 2$. Then we have

$$\begin{aligned}\rho_{n+2}^2 &= \rho_n^2(1 + 2\gamma\cos\theta + \gamma^2)(1 - 2\gamma\cos\theta + \gamma^2)\\ &= \rho_n^2(1 - 2\gamma^2\cos 2\theta + \gamma^4).\end{aligned}$$

This shows that $\rho_{n+2}/\rho_n < 1$ if and only if (5) holds. Thus, if (5) holds, then ρ_n tends to 0 as $n \to \infty$, whenever $k \not\equiv 0 \pmod 4$. Since

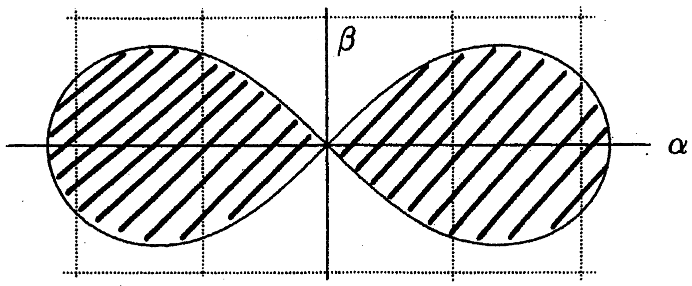
$$\sup_{t\in[n,n+1]} \|\mathbf{x}(t)\| = \sup_{t\in[n,n+1]} \|\mathbf{u}_n(t)\| \le \max\{\rho_n, \rho_{n+1}\},$$

we can conclude that if (5) holds, then the system (3) is uniformly asymptotically stable. It is easy to show that if the system (3) is uniformly asymptotically stable then ρ_n tends to 0 as $n \to \infty$ and hence (5) holds. Thus the proof is now completed. □

The following theorem is an immediate consequence of Proposition 1.

Theorem 1. *Let $c = \int_0^1 p(t)dt$, and assume that $\int_0^\omega p(t)dt = 0$ and $k \not\equiv 0$ (mod 4). Then the system (1) is uniformly asymptotically stable if and only if*

$$0 < (\alpha^2 + \beta^2)^2 < \frac{2}{c^2}(\alpha^2 - \beta^2).$$



Proof. Since $q(t) = rp(t)$, it is trivial that (5) is equivalent to

$$0 < r^2c^2 < 2\cos 2\theta$$

or

$$0 < r^4 < \frac{2}{c^2}r^2\cos 2\theta.$$

Therefore we can arrive at the conclusion of this theorem. □

For the case of $\int_0^\omega p(t)dt = 0$ but $k = 0$ (mod 4), we obtain the following result.

Theorem 2. *Assume that $\int_0^\omega p(t)dt = 0$ and $k = 4m$ for some positive integer m. Then every solution of (1) is ω-periodic for $t \ge N$, where N denotes the minimal integer not less than initial time t_0 of the solution.*

Proof. Since $\int_0^\omega p(t) = 0$, periodicity of $p(t)$ implies

$$\int_t^{t+\omega} p(s)ds = 0,$$

so that $q(t)$ also satisfies

$$\int_t^{t+\omega} q(s)ds = 0.$$

Now let $\mathbf{x}(t)$ be a solution of (1). Then it follows from (4) that

$$\mathbf{u}_n(t+\omega) = \mathbf{u}_n(t)$$

and hence

$$\mathbf{x}(t+\omega) = \mathbf{x}(t), \tag{6}$$

whenever $n \le t < t+\omega \le n+1$. On the other hand, since $k = 4m$, ω satisfies $m\omega = 1$. This, together with (6), implies

$$\mathbf{x}(n) = \mathbf{x}(N) \quad \text{or} \quad \rho_n = \rho_N$$

for any integer $n > N$. Hence for each $\mathbf{u}_n(t)$, $\mathbf{u}(t) = \mathbf{u}_n(n+t)$ is unique solution on $[0,1)$ of the initial value problem

$$\dot{\mathbf{u}}(t) = q(t)\begin{pmatrix}\rho_N \\ 0\end{pmatrix}, \quad \mathbf{u}(0) = \rho_N\begin{pmatrix}\cos\theta \\ -\sin\theta\end{pmatrix},$$

which yields

$$\mathbf{u}_n(n+t) = \mathbf{u}_N(N+t)$$

and so

$$\mathbf{x}(n+t) = \mathbf{x}(N+t) \tag{7}$$

on $[0,1]$ for any integer $n > N$. Therefore (6) and (7) assert that

$$\mathbf{x}(t+\omega) = \mathbf{x}(t)$$

for all $t \ge N$. This completes the proof. □

References

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[3] S. Sakata and T. Hara, Dynamics of a linear differential system with piecewise constant argument, to appear in *Dynam. Contin. Discrete Impuls. Systems.*